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BALANCING VECTORS IN THE MAX NORM

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Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be vectors in $\mathbf{R}^n$ of max norm at most one. It is proven that there exists a choice of signs for which all partial sums have max norm at most $Kn^{1/2}$. It is further shown that such a choice of signs must be anticipatory—there is no way to choose the i-th sign without knowledge of $\mathbf{v}_j$ for $j>i$.

Notation. Throughout this paper $\|\cdot\|$ denotes the max, or L^∞, norm. That is, if $\mathbf{x}=(x_1, \dots, x_n)\in\mathbf{R}^n$ then $\|\mathbf{x}\|=\max|x_i|$.

0. Introduction

We shall discuss the following three Questions, each of which involves the balancing of vectors in the max norm. In all cases we consider K as an appropriately large absolute constant.

Question 1. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_n\in\mathbf{R}^n$, $\|\mathbf{v}_i\|\leqq 1$. *Do there exist* $\varepsilon_1, \dots, \varepsilon_n\in\{-1, +1\}$ *so that*

$$\|\varepsilon_1\mathbf{v}_1+\dots+\varepsilon_n\mathbf{v}_n\| \leqq Kn^{1/2}?$$

Question 2. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_n\in\mathbf{R}^n$, $\|\mathbf{v}_i\|\leqq 1$. *Do there exist* $\varepsilon_1, \dots, \varepsilon_n\in\{-1, +1\}$ *so that*

$$\|\varepsilon_1\mathbf{v}_1+\dots+\varepsilon_t\mathbf{v}_t\| \leqq Kn^{1/2}$$

for all t, $1\leqq t\leqq n$?

Question 3. *Consider the following n-round perfect information game between two players henceforth known as Pusher and Chooser. A vector* $\mathbf{w}\in\mathbf{R}^n$, *called the position, is set at the start of the game to* $\mathbf{0}$. *On the i-th round Pusher selects a vector* $\mathbf{v}=\mathbf{v}_i\in\mathbf{R}^n$ *with* $\|\mathbf{v}\|\leqq 1$. *We call* $\mathbf{v}$ *the move. Chooser then resets the position* $\mathbf{w}$ *to either* $\mathbf{w}+\mathbf{v}$ *or* $\mathbf{w}-\mathbf{v}$. *With perfect play can Chooser assure that at the end of the n-th round* $\|\mathbf{w}\|\leqq Kn^{1/2}$?

Our three Questions form a progression. Clearly a positive answer to Question 2 implies a positive answer to Question 1. Now suppose the answer to Question 2

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were No with vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. In the game of Question 3 Pusher could make $\mathbf{v}_i$ his i-th move. Letting ε_i denote Chooser's i-th choice of sign, the position after t moves is given by $\mathbf{w}=\varepsilon_1\mathbf{v}_1+\ldots+\varepsilon_t\mathbf{v}_t$. Chooser would be forced to make $\|\mathbf{w}\|>Kn^{1/2}$ after some move t. At this point Pusher would change his strategy and make all future moves $\mathbf{v}=0$. Thus the answer to Question 3 would be No. We are indebted to Michael Todd (Cornell) for posing the intermediate second Question.

We have previously shown ([2]) that the answer to the third Question (and hence to all three) is Yes if $Kn^{1/2}$ is replaced by $cn^{1/2}(\ln n)^{1/2}$. The essential problem, then, is one of removing the $(\ln n)^{1/2}$ factor. In a probabilistic sense the factor $n^{1/2}$ represents a single standard deviation. We think of these Questions as asking whether or not one can remain within a fixed number of standard deviations of the zero mean.

In our recent paper ([3]) we were able to meld the probabilistic method and the pigeonhole principle and show that the answer to Question 1 is Yes. Our current work, while self-contained, may be regarded as a sequel. In section 1, using the techniques developed in [3], we give a positive answer to Question 2. Somewhat surprisingly, we show in section 2 that the answer to Question 3 is No. Here the methods are entirely different. We give a simple strategy (though the analysis is a bit complex) for Pusher that render Chooser helpless and unable to keep the norm of the position vector within the desired bound. In the final section we examine our three Questions when the number of vectors is allowed to be arbitrarily large but the dimension remains fixed at n.

The argument in [3] giving the existence of $\varepsilon_1, \ldots, \varepsilon_n$ satisfying Question 1 does not give a good (i.e. polynomial time) algorithm for finding the ε's and it is not known whether such an algorithm exists. Our negative answer to Question 3, while not resolving this question, tells us roughly that the ε's cannot be found locally—to determine ε_i one must examine also the "later" $\mathbf{v}_j$.

1. Bounding partial sums

While the results of this section are self-contained we note that Lemmas 1.1 and 1.3 follow quite closely the methods of Lemmas 4 and 6 respectively of our previous work ([3]). Our object in this section is to give a Yes answer to Question 2.

Lemma 1.1. *There are absolute constants K, c with $c<1$ so that the following holds for n sufficiently large. Assume $|a_{ij}|\leqq 1$, $1\leqq i, j\leqq n$. Then there exist $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 0, +1\}$ such that*

(i) *at most cn of the ε_i are zero*

(ii) *all $|s_{it}|\leqq Kn^{1/2}$*

where we let s_{it} denote the partial sum $(s_{i0}=0)$

$$s_{it} = \varepsilon_1 a_{i1}+\ldots+\varepsilon_t a_{it}$$

Setting $\mathbf{v}_i=(a_{1i}, \ldots, a_{ni})$, the i-th column vector of the matrix (a_{ij}), Lemma 1.1 gives ε's so that all $\|\varepsilon_1\mathbf{v}_1+\ldots+\varepsilon_t\mathbf{v}_t\|\leqq Kn^{1/2}$.

Proof. We shall define a map T with domain $\{-1, +1\}^n$ by $T(\varepsilon_1, \ldots, \varepsilon_n)=(\mathbf{b}_1, \ldots, \mathbf{b}_n)$ where the $\mathbf{b}_i$ will be described below. The ε's generate partial sums s_{it}. For each i let

T_i denote the least positive integer so that $|j-k|\leqq n/T_i$ implies $|s_{ij}-s_{ik}|\leqq Kn^{1/2}$ (T_i gives a measure of the variation of the partial sums $s_{i1}, \ldots, s_{in}$. In particular, if $T_i=1$ then these partial sums don't stray too far). For $1\leqq u\leqq T_i$ we set $q=un/T_i$ and

$$b_{iu} \text{ equal the nearest integer to } s_{iq}/2Kn^{1/2}.$$

We define $\mathbf{b}_i$ to be the vector $\mathbf{b}_i=(T_i\colon b_{i1}, \ldots, b_{iT_i})$. (The value q must actually be integral. We suppress the technical modifications needed, which do not affect the asymptotic analysis.) The definition of T_i insures that $|s_{ij}-s_{ik}|\leqq Kn^{1/2}$ when $j=(u+1)n/T_i$ and $k=un/T_i$ so that $|b_{i,u+1}-b_{iu}|\leqq 1$. Also $b_{i1}=0$. There is only one possible $\mathbf{b}_i$ with $T_i=1$, namely $(1\colon 0)$. There are at most 3^{T-1} possible $\mathbf{b}_i$ with a given value $T=T_i$ since given b_{iu} there are only three possible choices for $b_{i,u+1}$.

We now jump to the "end" of the proof and assume we have found $\varepsilon', \varepsilon''\in\{-1, +1\}^n$ for which $T(\varepsilon')=T(\varepsilon'')$. We then set $\varepsilon=(\varepsilon_1, \ldots, \varepsilon_n)=(\varepsilon'-\varepsilon'')/2$ so that $\varepsilon\in\{-1, 0, +1\}^n$. Let i, t be arbitrary $(1\leqq i, t\leqq n)$ and consider the partial sums

$$s'_{it} = \varepsilon'_1 a_{i1}+\ldots+\varepsilon'_t a_{it}$$

$$s''_{it} = \varepsilon''_1 a_{i1}+\ldots+\varepsilon''_t a_{it}$$

$$s_{it} = \varepsilon_1 a_{i1}+\ldots+\varepsilon_t a_{it} = (s'_{it}-s''_{it})/2$$

where $\varepsilon'_j, \varepsilon''_j$ are the j-th coordinates of $\varepsilon', \varepsilon''$ respectively. Under the map T, ε' and ε'' are mapped onto the same i-th coordinate, say $\mathbf{b}_i=(T_i\colon b_{i1}, \ldots, b_{iT_i})$. There exists u, $1\leqq u\leqq T_i$, such that, setting $q=un/T_i$, $|q-t|\leqq n/T_i$. Since $\varepsilon', \varepsilon''$ map to the same b_{iq} the values s'_{iq} and s''_{iq} lie on a common interval of length $2Kn^{1/2}$ so that $|s'_{iq}-s''_{iq}|\leqq 2Kn^{1/2}$. The definition of T_i insures that $|s'_{iq}-s'_{it}|\leqq Kn^{1/2}$ and $|s''_{iq}-s''_{it}|\leqq Kn^{1/2}$. By the triangle inequality $|s'_{it}-s''_{it}|\leqq 4Kn^{1/2}$ so that $|s_{it}|\leqq 2Kn^{1/2}$, condition (ii) (changing K to $2K$) of Lemma 1.

As our proof makes heavy use of probabilistic methods it will be convenient to prove a technical probability lemma at this point.

Definition. We shall call $\varepsilon_1, \ldots, \varepsilon_n$ *coin flips* if they are mutually independent random variables with $\Pr(\varepsilon_i=+1)=\Pr(\varepsilon_i=-1)=1/2$ for all i, $1\leqq i\leqq n$.

Lemma 1.2. *Assume* $|a_i|\leqq 1$, $1\leqq i\leqq n$. *Let* s_t *denote the partial sum* $s_t=\varepsilon_1 a_1+\ldots+\varepsilon_t a_t$ *generated by coin flips* ε_i. *Let* E *be the event that* $|s_j-s_k|>Kn^{1/2}$ *for some* $0\leqq j<k\leqq n$ *with* $|j-k|\leqq n/T$. *Then*

$$\Pr(E) \leqq 4Te^{-K^2T/16}.$$

Surely this bound could be improved. We actually need only that if K is very large then $\Pr(E)$ is quite small when $T=1$ and rapidly approaches zero in T.

Proof. For $1\leqq u\leqq T$ let E_u be the event that, setting $q=un/T$, there exists j with $q\leqq j\leqq q+2n/T$ such that $|s_j-s_q|>Kn^{1/2}/2$. Suppose E occurs for some particular j, k. There is a u such that $q\leqq j, k\leqq q+2n/T$. By the triangle inequality either $|s_q-s_j|$ or $|s_q-s_k|$ is at least $Kn^{1/2}/2$ so that E_u holds. The Reflection Principle implies

$$\Pr(E_u) \leqq 2\Pr(|s_{q+2n/T}-s_q|) > Kn^{1/2}/2.$$

We require the basic fact (see, e.g., [3]) that for any $a_1, \ldots, a_m$ with all $|a_i| \leqq 1$, coin flips ε_i, and any $\lambda > 0$

$$\Pr(|\varepsilon_1 a_1 + \ldots + \varepsilon_m a_m| > \lambda m^{1/2}) < 2e^{-\lambda^2/2}.$$

Applying this inequality to the sum of $2n/T$ terms above $(m = 2n/T, \lambda = K(T/8)^{1/2})$

$$\Pr(E_u) \leqq 2 \cdot 2e^{-K^2T/16}.$$

The event E is contained in the disjunction of the events E_u so Pr (E) is at most the sum of the T values $\Pr(E_u)$ and Lemma 1.2 is proven. ∎

We continue the proof of Lemma 1.1. We define a subset B of the range of T by (letting T_i denote the first coordinate of b_i)

$$B = \{(\mathrm{b}_1, \ldots, \mathrm{b}_n)\colon \text{at most } n(4Te^{-K^2T/16})2^{T+1} \text{ of the } T_i \text{ are greater than } T \text{ for every } T \geqq 1\}.$$

Claim 1. $T^{-1}(B) \geqq 2^{n-1}$.

Proof. Consider T_i generated by coin flips $\varepsilon_1, \ldots, \varepsilon_n$. By Lemma 2

$$\Pr(T_i > T) < 4Te^{-K^2T/16}.$$

The expected value of the number of i with $T_i > T$ is therefore at most $n(4T \exp(-K^2T/16))$. (Note that we have not used, indeed do not have, any independence of the variables T_i.) Any nonnegative random variable is more than α times its expected value at most $1/\alpha$ of the time. Hence the probability that more than $n(4T \exp(-K^2T/16))2^{T+1}$ of the T_i are greater than T is at most $1/2^{T+1}$. Summing over all $T \geqq 1$ $\Pr((\mathbf{b}_1, \ldots, \mathbf{b}_n) \notin B) \leqq \sum 1/2^{T+1} = 1/2$. With probability at least 1/2 coin flips $\varepsilon_1, \ldots, \varepsilon_n$ generate $(\mathbf{b}_1, \ldots, \mathbf{b}_n) \in B$. The probability of an event is simply the number of $(\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, +1\}^n$ for which the event holds divided by 2^n. ∎

Claim 2. *For K a sufficiently large absolute constant,* $|B| \leqq 2^{\delta n}$ *with* $\delta < 1$.

Proof. Set $\beta = 4T \exp(-K^2T/16)2^{T+1}$ for convenience. Then

$$|B| \leqq \prod_{T=1}^{\infty} \left[\sum_{s=0}^{\beta n} \binom{n}{s} \right] (3^T)^{\beta n}.$$

Here the first term is a bound on the number of ways of choosing $\{i\colon T_i = T+1\}$ and the second term is the 3^T possible b_i for each of the at most βn indices i. We bound

$$\sum_{s=0}^{\beta n} \binom{n}{s} \leqq 2^{nH(\beta)}$$

where H is the Entropy Function

$$H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$$

so that $|B| \leqq 2^{\delta n}$ where

$$\delta = \sum_{T=1}^{\infty} H(\beta) + T\beta \log_2 3.$$

For K large a simple analysis (as done in our previous work) shows that this infinite sum is dominated by the $T=1$ and the δ approaches zero as K approaches infinity. We require only a particular value of K for which $\delta<1$. We may take, for example, $K=10$ and $\delta=.21$. ∎

Combining the two claims, T maps at least 2^{n-1} elements of $\{-1, +1\}^n$ into at most $2^{\delta n}$ values. By the Pigeonhole Principle there is a set $\mathcal{A}\subset\{-1, +1\}^n$ on which T is constant with $|\mathcal{A}|\geqq 2^{n(1-\delta)-1}$. Let α be an absolute constant with $H(\alpha)<1-\delta$ (with $\delta=.21$, $\alpha=.23$) and let n be so large that $n(1-\delta)-1>nH(\alpha)$. Let ε' be an arbitrarily selected element of $\mathcal{A}$. The number of $\varepsilon''\in\{-1, +1\}^n$ which differ from ε' in at most αn coordinates is at most $2^{nH(\alpha)}$. As this is smaller than $|\mathcal{A}|$ there exists $\varepsilon''\in\mathcal{A}$ differing from ε' in more than αn coordinates. (In our earlier paper we used a powerful theorem of D. Kleitman on the diameter of subsets of the Hamming cube to find a considerably larger value of α. Kleitman's Theorem would be useful in reducing the value of the constant K but is not necessary for proving the existence of the constant.) Fix ε' and ε'' and set $\varepsilon=(\varepsilon'-\varepsilon'')/2$. Condition (i) of Lemma 1.1 is met with $c=1-\alpha$ (i.e. $c=.77$) and Condition (ii) by our earlier remarks ($K=20$) so the Lemma is proven. ∎

The following result, being a straightforward generalization of Lemma 1.1, is proven only in outline form.

Lemma 1.3. *There are absolute constants K, c with $c<1$ so that the following holds for all $r\leqq n$ with r sufficiently large. Assume $|a_{ij}|\leqq 1$, $1\leqq i\leqq n$, $1\leqq j\leqq r$. Then there exist $\varepsilon_1, \dots, \varepsilon_r\in\{-1, 0, +1\}$ such that*

(i) *at most cr of the ε_i are zero*

(ii) *all $|s_{it}|<Kr^{1/2}(\ln(2n/r))^{1/2}$*

where s_{it} is the partial sum defined in Lemma 1.1.

Setting $\mathbf{v}_i=(a_{1i}, \dots, a_{ni})$, Lemma 1.3 gives ε's so that all $\|\varepsilon_1\mathbf{v}_1+\dots+\varepsilon_t\mathbf{v}_t\|<Kr^{1/2}(\ln(2n/r))^{1/2}$.

Proof (Outline). Define T with domain $\{-1, +1\}^r$ by $T(\varepsilon_1, \dots, \varepsilon_r)=(\mathbf{b}_1, \dots, \mathbf{b}_n)$ where the $\mathbf{b}_i$ will be described below. The ε's generate partial sums s_{it}. For each i let T_i denote the least positive integer so that $|j-k|\leqq r/T_i$ implies $|s_{ij}-s_{ik}|\leqq\beta r^{1/2}$ where we set $\beta=K(\ln(2n/r))^{1/2}$ for notation convenience. For $1\leqq u\leqq T_i$ we set $q=ur/T_i$ and

b_{iu} equal the nearest integer to $s_{iq}/2\beta r^{1/2}$.

We define $\mathbf{b}_i$ to be the vector $\mathbf{b}_i=(T_i: b_{i1}, \dots, b_{iT_i})$. As before, there is only one $\mathbf{b}_i$ with $T_i=1$ and at most 3^{T-1} possible $\mathbf{b}_i$ with $T_i=T$. We set

$$B=\{(\mathbf{b}_1, \dots, \mathbf{b}_n)\colon \text{ at most } n(4Te^{-T\beta^2/16})2^{T+1} \text{ of the } T_i \text{ are greater than } T \text{ for every } T\geqq 1\}.$$

Lemma 2 implies that, with T_i generated from coin flips $\varepsilon_1, \dots, \varepsilon_r$,

$$\Pr(T_i>T)<4Te^{-T\beta^2/16}$$

so, as before, $|T^{-1}(B)|\geqq 2^{r-1}$. A calculation gives also that $|B|\leqq 2^{\delta r}$ where, for K a sufficiently large absolute constant, $\delta<1$. (In this calculation the logarithmic term comes into play. The lead term, when $T=1$, is at most $2^{nH(\alpha)}$ where $\alpha=4e^{-\beta^2/16}2^2=16(r/2n)^{K^2/16}$. For K large $H(\alpha)<.01(r/n)$ for any $n\geqq r$ with r sufficiently large.) By the Pigeonhole Principle there exists $\mathcal{A}\subset\{-1,+1\}^r$, $|\mathcal{A}|\geqq 2^{r(1-\delta)-1}$, on which T is constant and the proof may be concluded as before. ∎

Theorem 1.4. *The answer to Question 2 is Yes.*

Proof. Let $\mathbf{v}_1,\ldots,\mathbf{v}_n\in\mathbf{R}^n$, $\|\mathbf{v}_i\|\leqq 1$, be given. By Lemma 1 there exist $\varepsilon_1,\ldots,\varepsilon_n\in\{-1,0,+1\}$ with at most cn of them equal to zero so that all partial sums w satisfy $\|\mathbf{w}\|\leqq Kn^{1/2}$. If ε_i is not zero then that will be the final value of ε_i. Apply Lemma 1.3 (of which Lemma 1.1 is really only a special case) to the $m\leqq cn$ vectors $\mathbf{v}_i$ whose corresponding ε_i is zero, retaining the ordering on these vectors given by their indices. A value $\varepsilon_i=+1$ or -1 is then given to a positive proportion of the coefficients, leaving at most $cm\leqq c^2n$ vectors with coefficient ε_i still zero. Iterate this process, finding nonzero ε's for a positive proportion of the vectors at each iteration, until the number of vectors remaining is too small for Lemma 1.3 to apply. These vectors are bounded in number by an absolute constant so we may choose their ε's arbitrarily and the asymptotic results will not be affected. Now consider any partial sum $\varepsilon_1\mathbf{v}_1+\ldots+\varepsilon_t\mathbf{v}_t$ from the original sequence and observe that it may be broken up into the partial sums found at each iteration. On the u-th iteration there are $r\leqq c^un$ variables ($u=0$ is the initial step) so the partial sums are then bounded by $Kr^{1/2}(\ln(2n/r))^{1/2}\leqq Kn^{1/2}c^{u/2}\ln(2c^{-u})$. Hence the full partial sum has norm bounded by

$$\sum_{u=0}^{\infty} Kn^{1/2}c^{u/2}\ln(2c^{-u})\leqq K'n^{1/2}$$

where

$$K'=K\sum_{u=0}^{\infty}c^{u/2}\ln(2c^{-u})$$

is an absolute constant. ∎

2. The balancing game

In this section we examine the game between Pusher and Chooser defined in Question 3. It will be convenient to end the game with a payoff from Pusher to Chooser of $\|\mathbf{w}\|$ and then to let $G(n)$ denote the value of that game to Pusher. In this context, Question 3 asks if $G(n)=O(n^{1/2})$. In fact, we shall show

$$\big(1-o(1)\big)(n\ln n)^{1/2}\leqq G(n)\leqq\big(1+o(1)\big)(2n\ln n)^{1/2}.$$

We describe a simple strategy for Pusher that will achive this lower bound. Let $\mathbf{w}=(w_1,\ldots,w_n)$ be the current position and $\mathbf{v}=(v_1,\ldots,v_n)$ denote Pusher's move. Group the coordinates by their value w_i. Split the coordinates with any particular value into two equally sized groups, leaving one coordinate out if there are an odd number. Pusher then sets $v_i=+1$ for all i in one group, $v_i=-1$ for the i in the second group, and $v_i=0$ for the odd coordinate if there is one. For example, in position $\mathbf{w}=(0,0,0,1,2,2)$ Pusher would select move $\mathbf{v}=(+1,-1,0,0,+1,-1)$.

Analysis of this strategy is greatly simplified by noting that Chooser's choice of sign is irrelevant. Both $\mathbf{w}+\mathbf{v}$ and $\mathbf{w}-\mathbf{v}$ have the same coordinate values, only a permutation of the indices (unimportant for our purposes) distinguishes them. We define $F\colon \mathbf{R}^n \to \mathbf{R}^n$ by $F(\mathbf{w}) = \mathbf{w} + \mathbf{v}$ where $\mathbf{v}$ is the move of Pusher from position $\mathbf{w}$. Then $G(n) \geqq \|F^{(n)}(0)\|$.

It is more convenient to deal directly with the distribution of the coordinates. A function f with domain the integers is called the distribution function for $\mathbf{w}$ if for each u there are precisely $f(u)$ indices i such that $w_i = u$. We define a (non-linear) operator T on these distribution functions by

$$(Tf)(u) = \begin{cases} [f(u-1)/2] + [f(u+1)/] & \text{if } f(u) \text{ is even} \\ [f(u-1)/2] + [f(u+1)/2] + 1 & \text{if } f(u) \text{ is odd.} \end{cases}$$

The operator T mirrors F in that if $F(\mathbf{w}) = \mathbf{w}'$ and $\mathbf{w}$, $\mathbf{w}'$ have distribution functions f, f' respectively then $T(f) = f'$. We define a support function supp (f) as the maximal positive integer α such that either $f(\alpha)$ or $f(-\alpha)$ is nonzero. Let n denote the function defined by $n(0) = n$, $n(x) = 0$ for all $x \neq 0$ so that n is the distribution function of $0 \in \mathbf{R}^n$. Then

$$G(n) \geqq \|F^{(n)}(0)\| = \operatorname{supp}(T^n(n)). \tag{2.1}$$

Example: $n = 20$. We write f as a vector $(\ldots, f(-1), f(0), f(1), \ldots)$, the zeroth coordinate being central. Then $20 = (20)$,

$$T(20) = (10, 0, 10),\; T^2(20) = (5, 0, 10, 0, 5),$$

$$T^3(20) = (2, 1, 7, 0, 7, 1, 2),\; T^4(20) = (1, 0, 5, 1, 6, 1, 5, 0, 1),$$

$$T^5(20) = (1, 2, 1, 6, 0, 6, 1, 2, 1), \ldots, T^{20}(20) = (1, 1, 1, 1, 1, 3, 1, 2, 1, 3, 1, 1, 1, 1, 1)$$

so that $G(20) \geqq 7$. Chooser has a specific strategy in the game defined in Question 3 with $n = 20$ so that the final $\mathbf{w}$ has $\|\mathbf{w}\| \geqq 7$.

Let coin flips $\varepsilon_1, \ldots, \varepsilon_n$ (as defined in section 1) generate partial sums $s_1, \ldots, s_n$ with $s_t = \varepsilon_1 + \ldots + \varepsilon_t$ and set

$$M_n = \max_{1 \leqq t \leqq n} |s_t|$$

M_n is the maximal distance from the origin in a random walk of time n beginning at the origin in which each step is one unit, equally likely to the left or right.

Theorem 2.1. *Let α be a positive integer so that $n\Pr(M_n > \alpha) > 2\alpha + 1$. Then $G(n) \geqq \alpha$.*

A simple calculation, using the estimation

$$\Pr(M_n > \alpha) = e^{-(1+o(1))\alpha^2/2n}$$

gives the lower bound $G(n) \geqq (1 - o(1))(n \ln n)^{1/2}$ from this result. From (2) it suffices to show supp $(T^n(n)) > \alpha$.

Proof. Assume supp $(T^n(n)) \leqq \alpha$. Set $\Omega = \{-\alpha, -\alpha+1, \ldots, 0, \ldots, \alpha-1, \alpha\} \cup \{\infty\}$. We define a linear operator S on functions $f\colon \Omega \to \mathbf{R}$ by

$$(Sf)(x) = (f(x-1) + f(x+1))/2, \quad |x| < \alpha$$

$$(Sf)(\alpha) = f(\alpha - 1)/2 \quad \text{and} \quad (Sf)(-\alpha) = f(1-\alpha)/2$$

$$(Sf)(\infty) = f(\infty) + (f(\alpha)/2) + (f(-\alpha)/2).$$

S is the transition operator for the random walk on Ω in which each step is one unit, equally likely to the left or right, ∞ is considered both to the right of α and to the left of $-\alpha$ and is itself an absorbing barrier. Consider a particle beginning at position 0 at time 0. Its positions will be given by the partial sums $s_1, s_2, \dots$ defined earlier with the proviso that if $|s_i|=\alpha+1$ then the particle remains at ∞ for all time $j \geqq i$. The probability of the particle being at ∞ at time n is the probability that some $|s_i|>\alpha$, i.e. that $M_n>\alpha$. The particle's initial distribution is 1, it begins at position 0 with probability 1, so

$$(S^n 1)(\infty) = \Pr(M_n > \alpha).$$

Since S is a linear operator

$$(S^n n)(\infty) = n \Pr(M_n > \alpha). \tag{2}$$

The linear operator S will act as an approximation to T.

We have assumed $\operatorname{supp}(T^n(n)) \leqq \alpha$. It is simple to show that $\operatorname{supp}(Tf) \geqq \operatorname{supp}(f)$ for any f. Hence $\operatorname{supp}(T^i(n)) \leqq \alpha$ for $0 \leqq i \leqq n$.

We had defined T for functions on the integers. But we may consider $T^n(n)$ with T defined on Ω since the point ∞ never comes into play. That is,

$$T^n(n)(\infty) = 0. \tag{2.3}$$

We shall bound $(S^n - T^n)(n)(\infty)$. For $|t| \leqq \alpha$ define u_t as that function with $u_t(t)=1$, $u_t(s)=0$ for all $s \neq t$.

$$S^n - T^n = \sum_{i=0}^{n-1} S^{n-i}T - S^{n-i-1}T^{i+1} = \sum_{i=0}^{n-1} S^{n-i-1}(S-T)T^i.$$

For $0 \leqq i \leqq n-1$ set

$$T^i(n) = \sum_{|t| \leqq \alpha} a_{it} u_t.$$

Observe that S and T have "nearly" the same affect on $T^i(n)$, the only distinction being when a_{it} is odd when a single u_t is held fixed by T but "split" by S. More precisely,

$$(S-T)T^i(n) = \sum_{t \in A_i} Su_t - u_t$$

where we set A_i equal to the set of those t such that a_{it} is odd. Premultiplication by S^{n-i-1} and evaluation at ∞ yield

$$S^{n-i-1}(S-T)T^i(n)(\infty) = \sum_{t \in A_i} (S^{n-i}u_t)(\infty) - (S^{n-i-1}u_t)(\infty).$$

For any $f \geqq 0$, $(Sf)(\infty) \geqq f(\infty)$. (This was the somewhat technical reason S was chosen with absorbing barrier.) When $f = S^{n-i-1}u_t$ we see that the above addends are positive for all t so that

$$S^{n-i-1}(S-T)T^i(n)(\infty) \leqq \sum_{|t| \leqq \alpha} (S^{n-i}u_t)(\infty) - (S^{n-i-1}u_t)(\infty).$$

Summing from $i=0$ to $n-1$

$$(S^n-T^n)(n)(\infty) \leqq \sum_{i=0}^{n-1} \sum_{|t|\leqq\alpha} (S^{n-i}u_t)(\infty)-(S^{n-i-1}u_t)(\infty)$$
$$= \sum_{|t|\leqq\alpha} \sum_{i=0}^{n-1} (S^{n-i}u_t)(\infty)-(S^{n-i-1}u_t)(\infty)$$
$$= \sum_{|t|\leqq\alpha} (S^n u_t)(\infty)-u_t(\infty).$$

Since $S^n u_t$ is a probability distribution we bound each addend above by unity so that $(S^n-T^n)(n)(\infty)\leqq 2\alpha+1$ which, combined with (2.2), (2.3) yields $n \Pr(M_n>\alpha)\leqq 2\alpha+1$ proving our theorem in contrapositive form.

The asymptotic upper bound for $G(n)$ was given by Theorem 2 (with $k=n$) in [2]. Here we give a more precise result.

Consider the following gamblers game with an unusual twist. There is a single player with an initial stake of u. There are s rounds. On each round the player may bet any amount a with $0\leqq a\leqq 1$. (The player has "infinite credit" so that he may bet regardless of his current stake.) He either wins or loses the bet a with probability 1/2. That is, the stake u changes to either $u+a$ or $u-a$ with probability 1/2. This is sometimes referred to as a fair Red-Black game. He wins the game if his final stake v satisfies $|v|\geqq\alpha$. (One may imagine that the gambler is betting with play money and he wins a real dollar if and only if the final stake v satisfies $|v|\geqq\alpha$.) Let $W_\alpha(u, s)$ denote the probability that the gambler will win this game when he plays with his optimal strategy.

Theorem 2.2. *If* $W_\alpha(0, n)<n^{-1}$ *then* $G(n)<\alpha$.

When α is integral and the initial stake u is zero it is natural to conjecture that the optimal strategy is simply to bet 1 (the maximum) at every turn until one reaches a stake $v=\pm\alpha$ and then bet zero (i.e., quit) forevermore. If this conjecture is true then for α integral $W_\alpha(0, n)=\Pr(M_n>\alpha)$. In that case our Theorem could be rewritten "If α is a positive integer and $n \Pr(M_n\geqq\alpha)<1$ then $G(n)<\alpha$" which would provide a pleasant counterpoint to our lower bound. When the gambler's goal is to reach stake $v\geqq\alpha$—i.e., to win α—with α integral, initial stake 0, M. Klawe [4] has shown that the optimal strategy is indeed to bet 1 at every turn until the goal is reached and zero forevermore.

Proof. Fix α, n with $W_\alpha(0, n)<n^{-1}$. Chooser announces that on each of his turns he will flip a fair coin to decide whether $\varepsilon=+1$ or $\varepsilon=-1$. A strategy for Pusher produces a final position $\mathbf{w}=(w_1, \ldots, w_n)$ where the w_i are now random variables. Let E denote the event "$\|\mathbf{w}\|\geqq\alpha$" and, for $1\leqq i\leqq n$, let E_i denote the event "$|w_i|\geqq\alpha$". Our assumption is that for any strategy of Pusher $\Pr(E_i)<n^{-1}$. Since E is the disjunction of the events E_i, $\Pr(E)<n(n^{-1})=1$. There is no strategy for Pusher that assures $\|\mathbf{w}\|\geqq\alpha$. However, we are dealing with a perfect information game and hence there must exist pure strategies giving the value— i.e., if the value was at least α there would exist a strategy for Pusher such that $\Pr(E)=1$. Thus $G(n)<\alpha$. ∎

A modification of the above argument allows us to give a specific, though perhaps not easily calculable, strategy for Chooser that forces $\|\mathbf{w}\|$ to be less than

α. Let us agree to write W instead of W_α for notational convenience. The functions $W(u, s)$ may be defined inductively on s by

$$W(u, 0) = \begin{cases} 1 & \text{if } |u| \geqq \alpha \\ 0 & \text{otherwise} \end{cases}$$

$$W(u, s+1) = \max_{0 \leqq a \leqq 1} \left(W(u+a, s) + W(u-a, s)\right)/2.$$

Extend these functions to vectors by defining, for $\mathbf{w} = (w_1, \ldots, w_n)$

$$W(\mathbf{w}, s) = \sum_{i=1}^{n} W(w_i, s).$$

Let $\mathbf{a} = (a_1, \ldots, a_n)$, $\|\mathbf{a}\| \leqq 1$. For $1 \leqq i \leqq n$

$$W(w_i, s+1) \geqq \left(W(w_i + a_i, s) + W(w_i - a_i, s)\right)/2.$$

Summing this inequality $W(\mathbf{w}, s+1) \geqq \left(W(\mathbf{w}+\mathbf{a}, s) + W(\mathbf{w}-\mathbf{a}, s)\right)/2$. Chooser's strategy is to select a sign so that $W(\mathbf{w}, s)$ will be minimized, where $\mathbf{w}$ is the new position and s is the number of rounds remaining after the choice is made. Let $\mathbf{w}$ be the position with $s+1$ moves remaining, $\mathbf{a}$ the next move of Pusher, and $\mathbf{w}'$ the new position. Then

$$\begin{aligned} W(\mathbf{w}', s) &= \min\left(W(\mathbf{w}+\mathbf{a}, s), W(\mathbf{w}-\mathbf{a}, s)\right) \\ &\leqq \left(W(\mathbf{w}+\mathbf{a}, s) + W(\mathbf{w}-\mathbf{a}, s)\right)/2 \\ &\leqq W(\mathbf{w}, s+1). \end{aligned}$$

At the beginning of the game the position is $\mathbf{0}$ and there are n moves remaining and $W(\mathbf{0}, n) = nW(0, n) < 1$ by assumption. Hence at the end of the game, with position $\mathbf{w}$ and 0 moves remaining, $W(\mathbf{w}, 0) < 1$. But the game is now over, all $W(w_i, 0)$ are either zero or one, hence they must all be zero, all $|w_i| < \alpha$, $\|\mathbf{w}\| < \alpha$ as Chooser desired.

3. Many vectors

In this section we reconsider our Three Questions when the number of vectors m is allowed to be arbitrarily large while the dimension n remains fixed. This extension of Question 1 was resolved in [3] wherein we showed that for all $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbf{R}^n$, $\|\mathbf{v}_i\| \leqq 1$, there exist $\varepsilon_1, \ldots, \varepsilon_m \in \{-1, +1\}$ with $\|\varepsilon_1 \mathbf{v}_1 + \ldots + \varepsilon_m \mathbf{v}_m\| \leqq K n^{1/2}$. The absolute constant here has twice the value of the constant in the original formulation. For the extension of Question 3 the following simple strategy shows that Chooser cannot keep the norm $\|\mathbf{w}\|$ of the position within an absolute bound. Given a position $\mathbf{w}$ Pusher selects a move $\mathbf{v}$ with $\|\mathbf{v}\| = 1$ and $\mathbf{v}$ perpendicular to $\mathbf{w}$. Let $\mathbf{w}'$ denote the new position. Regardless of Chooser's choice of sign $|\mathbf{w}'|^2 = |\mathbf{w} \pm \mathbf{v}|^2 = |\mathbf{w}|^2 + |\mathbf{v}|^2 \geqq \geqq |\mathbf{w}|^2 + 1$ as $|\mathbf{v}| \geqq \|\mathbf{v}\| = 1$. (Here $|v|$ is the Euclidean norm of v.) After m rounds the position $\mathbf{w}$ satisfies $|\mathbf{w}|^2 \geqq m$ so that $\|\mathbf{w}\| \geqq |\mathbf{w}| n^{-1/2} \geqq (m/n)^{1/2}$ which tends to infinity. For the extension of Question 2, I. Bárány and V. S. Grünberg ([1], Theorem 3 with $C_i = \{\mathbf{v}_i, -\mathbf{v}_i\}$) have shown the following.

Theorem 3.1. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbf{R}^n$, $\|\mathbf{v}_i\| \leqq 1$. *Then there exist* $\varepsilon_1, \dots, \varepsilon_m \in \{-1, +1\}$ *so that* $\|\varepsilon_1\mathbf{v}_1 + \dots + \varepsilon_t\mathbf{v}_t\| \leqq 2n$ *for* $1 \leqq t \leqq m$. ∎

Thus the norms of the partial sums can indeed be bounded by a function independent of m. Bárány and Grünberg employ a clever modification of the linear algebra method. Remarkably, they prove their result when $\|\cdot\|$ is replaced by any norm in $\mathbf{R}^n$. It is unclear whether the bound may be improved for various specific norms and we conclude our paper with an open problem: Is it the case that for all $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbf{R}^n$, $\|v_i\| \leqq 1$, there exist $\varepsilon_1, \dots, \varepsilon_m \in \{-1, +1\}$ such that $\|\varepsilon_1\mathbf{v}_1 + \dots + \varepsilon_t\mathbf{v}_t\| \leqq Kn^{1/2}$ for all t, $1 \leqq t \leqq m$?

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